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Analytical solutions to simultaneously developing laminar flow inside parallel-plate channels

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Abstract—Simultaneous development of velocity and temperature distributions for laminar flow inside a parallel-plate channel is analytically studied, by adopting a linearization procedure for the velocity problem and solving the decoupled energy equation through the generalized integral transform technique. A complete solution is obtained within a wide range of the axial coordinate, from numerical evaluation of the integral transformed system of ordinary differential equations. In addition, approximate explicit solutions are provided for fast estimates in the context of applications. Several aspects are investigated, such as influence of transversal convection, effects of different velocity profiles, convergence of complete solution, and accuracy of approximate solutions.

INTRODUCTION

THE ANALYSIS of simultaneously developing laminar flows has been of great interest, as demonstrated by the vast literature available, in connection with a demand for more precise reference data by heat exchanger designers [1]. In the famous sourcebook of reference results [2], Shah and London provide an extensive list of contributions related to this class of problems, mostly in simple geometries such as circular tubes, parallel-plate channels, and annular ducts. More recent reviews [3, 4], complement such a list and provide an indication that most previous work has been directed to circular tube geometry and to employing some kind of purely numerical technique to approximately solve the associated flow and energy equations. A commonly used approach, since the pioneering work of Kays [5], is that of utilizing explicit expressions for the velocity components obtained from linearization of the axial momentum equation, such as the better known velocity profiles by Langhaar [6] and Sparrow *et al.* [7]. The decoupled energy equation is then numerically solved for the temperature distribution, eventually after neglection of the radial convection term [5, 8, 9]. For the specific situation of a parallel-plate channel, representative contributions include the integral method approach of Siegel and Sparrow [10], the finite-differences solutions of Hwang and Fan [11] and Mercer *et al.* [12], and the approximate analytical solution of Han [13].

The exact solution of internal forced convection problems for the case of fully developed velocity profiles has been recently reviewed and presented in a systematic way by Mikhailov and Özişik [14], through the use of the classical integral transform technique. This approach was then employed to produce highly accurate benchmark results for the extended Graetz problem [15], over a wide range of the dimensionless axial coordinates, after the related eigenvalue problem has been automatically and accurately solved by utilizing the also recently advanced sign-count method for Sturm–Liouville problems [14]. The approach described in ref. [14] is not, however, directly applicable to the present simultaneously developing flow problem, due to the non-transformable nature of the resulting energy equation, which includes non-separable velocity functions that depend on both the normal and axial variables. The ideas in the so-called generalized integral transform technique [16–24] can, however, be applied to this class of problems to produce fully converged numerical results, obtained from a complete solution of the resulting coupled system of ordinary differential equations for the integral transformed temperature distribution. Besides, approximate analytical solutions in explicit form are readily obtainable, which have essentially the same degree of complexity as the exact solution of the Graetz problem in refs. [14, 15]. The present note brings this extension to the generalized integral transform technique, by considering explicit velocity profiles obtained by Targ [14, 25], which are more easily computable than those more frequently employed in the literature [6, 7].

ANALYSIS

Simultaneously developing laminar flow of a Newtonian fluid within a parallel-plate channel is considered. Physical properties are assumed to be

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NOMENCLATURE

D_e hydraulic diameter of parallel-plate channel, $4r_w$
$h(z)$ heat transfer coefficient
k thermal conductivity of fluid
N truncation order of system (6)
N_i norm of problem (3)
$Nu(Z)$ local Nusselt number, $h(z)D_e/k$
$Nu_{av}(Z)$ average Nusselt number
Pr Prandtl number, ν/α
r, R normal or transversal coordinate [dimensional and dimensionless]
r_w half the spacing between parallel-plates
Re Reynolds number, u_0D_e/ν
$T(r, z)$ temperature distribution [dimensional]
T_e uniform inlet temperature
T_w prescribed wall temperature
$u(r, z)$, $U(R, Z)$ axial velocity components [dimensional and dimensionless]
u_0 inlet flow velocity
$v(r, z)$, $V(R, Z)$ normal velocity components [dimensional and dimensionless]
$W(R)$ weight function of problem (3)
z, Z axial coordinates [dimensional and dimensionless].

Greek symbols
α thermal diffusivity of fluid
α_n roots of transcendental equation (A4)
$\theta(R, Z)$ dimensionless temperature distribution
$\theta_r(R, Z)$ lowest order solution (L.O.S.)
μ_i eigenvalues of problem (3)
ν kinematic viscosity
$\psi_i(R)$ eigenfunctions of problem (3).

temperature independent and the effects of axial conduction, viscous dissipation and free convection are neglected. The decoupled flow problem is solved by a linearization procedure, as described in the Appendix, providing explicit expressions for the axial and normal components of the velocity field. The related energy equation is then written in dimensionless form as

$$U(R,Z)\frac{\partial\theta(R,Z)}{\partial Z}+V(R,Z)\frac{\partial\theta(R,Z)}{\partial R}=\frac{1}{Pr}\frac{\partial^2\theta(R,Z)}{\partial R^2},\quad \text{in } 0<R<1, Z>0 \quad (1a)$$

with inlet and boundary conditions, respectively

$$\theta(R,0)=1,\quad 0\leqslant R\leqslant 1 \quad (1b)$$

$$\left.\frac{\partial\theta(R,Z)}{\partial R}\right|_{R=0}=0;\quad \theta(1,Z)=0, Z>0 \quad (1c,d)$$

where various dimensionless groups are given by

$$R=\frac{r}{r_w};\quad Z=\frac{\nu z}{u_0r_w^2};\quad U(R,Z)=\frac{u(r,z)}{u_0};$$

$$V(R,Z)=\frac{v(r,z)r_w}{\nu};\quad \theta(R,Z)=\frac{T(r,z)-T_w}{T_e-T_w};$$

$$Pr=\frac{\nu}{\alpha}. \quad (2)$$

Following the formalism in the generalized integral transform technique, as applied to the solution of the diffusion problems with variable equation coefficients [17], the appropriate auxiliary problem is taken as

$$\frac{d^2\psi_i(R)}{dR^2}+\mu_i^2W(R)\psi_i(R)=0,\quad 0<R<1 \quad (3a)$$

with boundary conditions

$$\frac{d\psi_i(0)}{dR}=0;\quad \psi_i(1)=0 \quad (3b,c)$$

where $W(R)$ is some characteristic weight function to be discussed later, and the solution of problem (3) for the associated eigenvalues, μ_is, and eigenfunctions, $\psi_i(R)$, is assumed to be known at this point.

Problem (3) above allows the definition of the following integral transform pair:

$$\bar{\theta}_i(Z)=\int_0^1 W(R)\mathbb{K}_i(R)\theta(R,Z)\,dR,\ \text{transform} \quad (4a)$$

$$\theta(R,Z)=\sum_{i=1}^{\infty}\mathbb{K}_i(R)\bar{\theta}_i(Z),\ \text{inverse} \quad (4b)$$

where the symmetric kernel is given by

$$\mathbb{K}_i(R)=\frac{\psi_i(R)}{N_i^{1/2}} \quad (4c)$$

and the normalization integral is obtained from

$$N_i=\int_0^1 W(R)\psi_i^2(R)\,dR. \quad (4d)$$

Equation (1a) is now operated on with

$$\int_0^1 \mathbb{K}_i(R)\,dR$$

to yield, after substitution of the inverse formula (4b)

$$\sum_{j=1}^{\infty}\left[a_{ij}(Z)\frac{d\bar{\theta}_j(Z)}{dZ}+b^*_{ij}(Z)\bar{\theta}_j(Z)\right]=-\frac{\mu_i^2}{Pr}\bar{\theta}_i(Z) \quad (5a)$$

where

$$a_{ij}(Z)=\int_0^1 U(R,Z)\mathbb{K}_i(R)\mathbb{K}_j(R)\,dR \quad (5b)$$

$$b^*_{ij}(Z)=\int_0^1 V(R,Z)\mathbb{K}_i(R)\frac{d\mathbb{K}_j(R)}{dR}\,dR. \quad (5c)$$

Similarly, the transformed inlet condition is obtained through the operator

$$\int_0^1 W(R)\mathbb{K}_i(R)\,dR$$

to provide

$$\bar{\theta}_i(0)=\bar{f}_i=\int_0^1 W(R)\mathbb{K}_i(R)\,dR. \quad (5d)$$

Equations (5) form an infinite system of coupled O.D.E.s for the transformed temperatures, $\bar{\theta}_i$s, which is to be truncated at the Nth row and column for computation purposes, with N a sufficiently large order for convergence to a certain prescribed accuracy. Formal aspects behind this truncation process, including an analysis of sufficient conditions for convergence and a priori error bounds, have been considered in refs. [17, 24] and are therefore not repeated here. The truncated version of system (5) is rewritten in matrix form as

$$\mathbf{A}(Z)\underset{\sim}{y}'(Z)+\mathbf{B}(Z)\underset{\sim}{y}(Z)=0,\quad Z>0 \quad (6a)$$

$$\underset{\sim}{y}(0)=\underset{\sim}{f} \quad (6b)$$

where

$$\underset{\sim}{y}(Z)=\{\bar{\theta}_1(Z),\ldots,\bar{\theta}_N(Z)\}^{T} \quad (6c)$$

$\mathbf{A}(Z)=\{a_{ij}(Z)\}$, is a symmetric $N\times N$ matrix (6d)

$$\mathbf{B}(Z)=\{b_{ij}(Z)\},\quad b_{ij}(Z)=\delta_{ij}\frac{\mu_i^2}{Pr}+b^*_{ij}(Z) \quad (6e)$$

and

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases};\quad \underset{\sim}{f}=\{\bar{f}_1,\ldots,\bar{f}_N\}^{T}. \quad (7a,b)$$

Since system (6) is likely to yield a stiff O.D.E. system, specially for increasing order N, due to the relatively different decay rates of the solution vector components, $y_i(Z)$, subroutine DIVPAG from the IMSL package [26], which implements the well-known Gear method, is in general preferred. Once the transformed temperatures have been numerically evaluated, the inverse formula (4b) is invoked to provide the complete temperature distribution, $\theta(R,Z)$, at any point within the duct cross-section.

Approximate solution

The approach just presented is an interesting alternative to purely numerical solutions, yielding highly accurate fully converged results. It would be extremely convenient, however, to also have explicit expressions available, though approximate, that could bring down the computational effort and allow for parameter and asymptotic inspections without requiring a complete numerical solution of the problem. This approach was in fact originally developed as an attempt to find approximate analytical solutions to classes of non-transformable problems, and later advanced to become a modern hybrid numerical–analytical computational tool [24]. Therefore, it allows for the establishment of explicit expressions, such as the so-called *lowest order solution* [17, 24], based on neglecting the contribution of non-diagonal elements in the coefficient matrices; the resulting decoupled O.D.E. system is then readily solved in explicit form. For the present application, such an approximate solution can be obtained through the following decoupled system:

$$\frac{d\bar{\theta}_{r,i}(Z)}{dZ}+c^*_{ii}(Z)\bar{\theta}_{r,i}(Z)=0,\quad Z>0 \quad (8a)$$

$$\bar{\theta}_{r,i}(0)=\bar{f}_i,\quad i=1,2,\ldots \quad (8b)$$

where

$$c^*_{ii}(Z)=\frac{\frac{\mu_i^2}{Pr}+b^*_{ii}(Z)}{a_{ii}(Z)} \quad (8c)$$

which is readily solved to yield the explicit formula after the inversion, equation (4b), is recalled

$$\theta_r(R,Z)=\sum_{i=1}^{\infty}\mathbb{K}_i(R)\bar{f}_i\exp\left\{-\int_0^Z c^*_{ii}(Z')\,dZ'\right\}. \quad (8d)$$

The numerical results obtained from equation (8d) above will be more or less accurate depending on the relative magnitudes of non-diagonal elements in matrices $\mathbf{A}(Z)$ and $\mathbf{B}(Z)$, which might be governed by the values of the axial coordinate, Z, and Prandtl number, Pr. Clearly, an appropriate choice of eigenvalue problem, will play an important role in improving the approximate solution within certain ranges of the governing parameters, Z and Pr, as will be discussed later.

Heat transfer quantities

Once the temperature profiles have been analytically obtained, quantities of practical interest in heat exchanger design can be obtained from their definitions, such as fluid bulk temperature along the duct length

$$\theta_{av}(Z)=\int_0^1 U(R,Z)\theta(R,Z)\,dR. \quad (9)$$

Also of interest is the local Nusselt number, evaluated from

$$Nu_1(Z) = \frac{-4\dfrac{\partial\theta(1,Z)}{\partial R}}{\theta_{av}(Z)} \tag{10a}$$

or alternatively from the heat balance equation

$$Nu_2(Z) = \frac{-4Pr\dfrac{d\theta_{av}(Z)}{\partial Z}}{\theta_{av}(Z)} \tag{10b}$$

which is obtained from integration of the energy equation over the channel cross-section, and the resulting expression for the dimensionless heat flux substituted in equation (10a) above.

The average Nusselt number is obtained from

$$Nu_{av}(Z) = \frac{1}{Z}\int_0^Z Nu(Z)\,dZ \tag{11a}$$

which by employing the alternative expression (10b) for the local Nusselt number, becomes

$$Nu_{av,2}(Z) = -4Pr\ln(\theta_{av}(Z))/Z. \tag{11b}$$

For fully converged results in the eigenfunction expansions, expressions (10a) and (10b) should yield identical numbers, and their comparison can be utilized as a convergence verification. The same can be said about equations (11a) and (11b) for the average Nusselt numbers.

RESULTS AND DISCUSSION

Numerical results were obtained over a wide range of the dimensionless axial coordinate ($Z \geqslant 10^{-5}$) and for different Prandtl numbers that cover the range of practical interest for the present formulation. The complete solution of system (6) was obtained with $N \leqslant 80$ to observe the convergence behaviour and in most situations reported, by utilizing Targ's velocity profiles (Appendix) and through the representative and simple choice of eigenvalue problem with $W(R) \equiv 1$. The choice of $W(R) = 1$ provides an exact decoupled solution for the plug flow situation ($Pr \rightarrow 0$), while a second characteristic choice of the weight function could be $W(R) = 1-R^2$, which yields a Graetz-type eigenvalue problem [14, 15] that decouples the system for fully developed flow conditions ($Pr \rightarrow \infty$). Although the convergence behaviour of the complete solution is not markedly affected by the choice of auxiliary problem, the relative accuracy of the lowest order solution is particularly influenced, as will be discussed later.

Table 1 illustrates the convergence behaviour of the dimensionless average temperature computed from the complete solution for different truncation orders (N) and for the two Prandtl numbers, $Pr = 0.72$ and 10.0. Clearly, convergence is achieved with a reasonably small number of coupled equations in system (6), and requiring an increasing N as Z is decreased. No significant effect of the Prandtl number on convergence rates could be observed, as noticeable for the two cases reported in Table 1. Figures 1(a) and (b) show the convergence behaviour of the average Nusselt numbers ($Pr = 0.72$), as computed from equations (11a) and (11b), respectively. Although for fully converged results the curves for $Nu_{av,1}(Z)$ and $Nu_{av,2}(Z)$ merge together, the curves for different values of N indicate the improved convergence rates provided by the heat balance equation, which is due to a faster convergence of the average temperature expressions over those for the temperature derivative at the wall.

Figure 2 shows a comparison of average Nusselt numbers obtained in the present work by utilizing approximate explicit velocity profiles, and reference results obtained from a finite-difference solution of the complete non-linear flow problem and the corresponding decoupled energy equation, available in ref. [2]. The practically coincident curves 2 and 3 in both figures, indicate the convergence of the analytical solution in the range of Z considered, since the numerical values of the Nusselt numbers as computed from two different expressions, equations (11a) and (11b), are in agreement. The adoption of Targ's velocity profiles results in reasonably accurate results, with increasing deviations as Z is decreased for both $Pr = 0.72$ and 10.0. For instance, around $Z = 10^{-4}$, relative errors are of the order of 13% for $Pr = 0.72$ and 8% for $Pr = 10.0$, and drop down sharply as the velocity field develops. The choice of the more general linearization procedure proposed by Sparrow *et al.* [7] does not significantly change this picture, as far as the average Nusselt number is concerned, as demonstrated in Fig. 3. For example, at $Z \simeq 10^{-4}$, the adoption of the approximate velocity profiles of Sparrow *et al.* [7], brings the average Nusselt number down by about 3% only, with a considerable increase in computational involvement, especially when including the normal convection term, with respect to Targ's profiles [25], which in terms of heat transfer quantities appears to be sufficiently accurate for most practical purposes.

The effects of neglecting the normal convection contribution in the energy equation are investigated through Fig. 4 for the local Nusselt number distributions. As expected and previously discussed [8], for both $Pr = 0.72$ and 10, the results without the normal convection term ($V = 0$) overestimate the heat transfer coefficient along the channel, especially at regions close to the inlet section. Certainly, the consideration of normal convection added to the normal diffusion effect, tends to make the temperature profiles less steep than in the approximate situation of $V = 0$. While the average temperature is not dramatically affected, the temperature derivative at the wall is sufficiently altered to reduce the local Nusselt number by about 19%, at $Z = 10^{-4}$, for $Pr = 0.72$, and by 20%, for $Pr = 10$. Therefore, previously reported numerical results based on neglecting normal con-

Table 1. Convergence of dimensionless average temperature for complete solution of system (6)

	Pr = 0.72				N	Pr = 10			
Z^*	10	20	35	50		5	20	35	50
5×10^{-4}	0.9272	0.9280	0.9283	0.9281		0.9389	0.9465	0.9470	0.9468
1×10^{-3}	0.8964	0.8970	0.8972	0.8971		0.9150	0.9209	0.9213	0.9211
2×10^{-3}	0.8515	0.8520	0.8522	0.8521		0.8760	0.8805	0.8809	0.8807
5×10^{-3}	0.7582	0.7586	0.7587	0.7586		0.7854	0.7886	0.7888	0.7888
1×10^{-2}	0.6456	0.6459	0.6460	0.6460		0.6695	0.6720	0.6721	0.6721
2×10^{-2}	0.4769	0.4771	0.4771	0.4771		0.4941	0.4958	0.4960	0.4959
5×10^{-2}	0.1930	0.1931	0.1931	0.1931		0.1999	0.2006	0.2006	0.2006
1×10^{-1}	0.04271	0.04273	0.04273	0.04273		0.04424	0.04439	0.04441	0.04440

$Z^* = (z/D_e)/(Re\ Pr)$.

vection sould be utilized with care at the inlet region [5, 8].

Figure 5 illustrates the relative accuracy of the explicit and quite straightforward lowest order solution from equation (8d), in terms of local Nusselt number distributions for $Pr = 0.72$. Curves 2 and 3 are, respectively, for the two different choices of auxiliary problem, with $W(R) = 1 - R^2$ and 1. Apparently, the eigenvalue problem with $W(R) = 1 - R^2$ produces, in overall behaviour, a less strongly coupled system, especially for increasing Z, when the fully developed region is approached. Relative errors are, around $Z = 10^{-4}$, of the order of 30% for the choice $W(R) = 1 - R^2$ and 40% for $W(R) = 1$, while around $Z = 10^{-3}$ the error in curve 2 drops down markedly to about 6% and stays around 32% for curve 3 ($W(R) = 1$). For increasing Z the error in curve 2 continues to drop until the asymptotic solution is reproduced exactly, whereas the error in curve 3 never improves over 7%. Similar trends were observed for

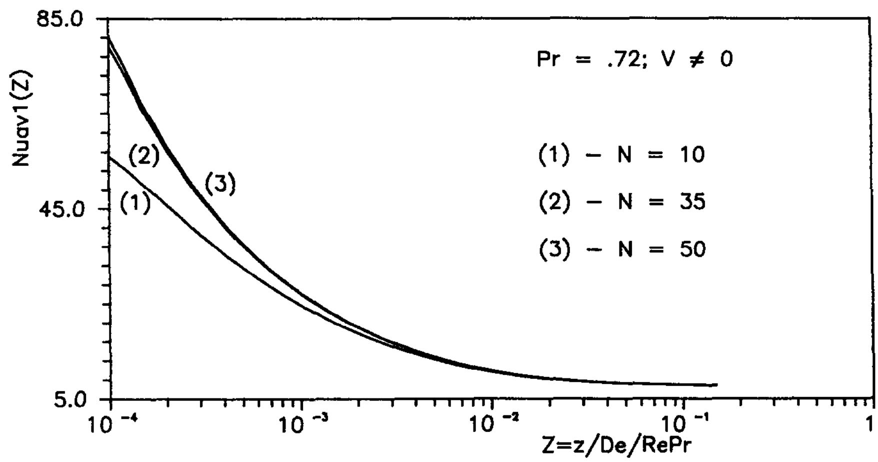

FIG. 1(a). Convergence of average Nusselt number as computed from equation (11a) ($Pr = 0.72$).

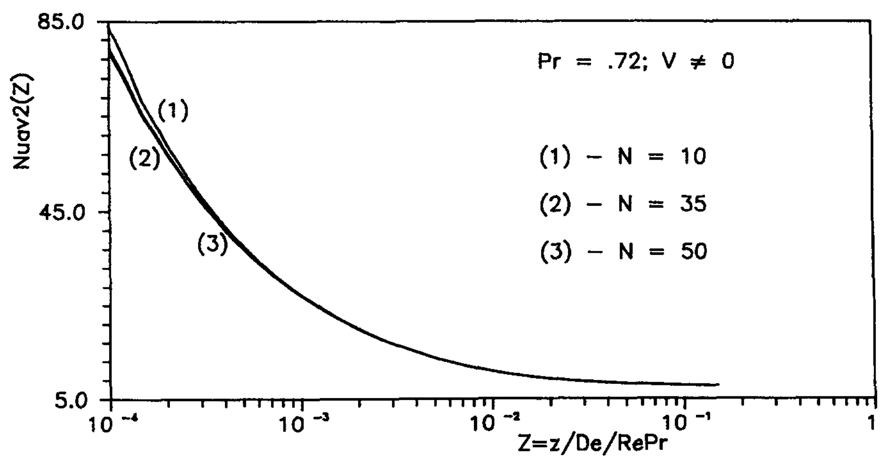

FIG. 1(b). Convergence of average Nusselt number as computed from equation (11b) ($Pr = 0.72$).

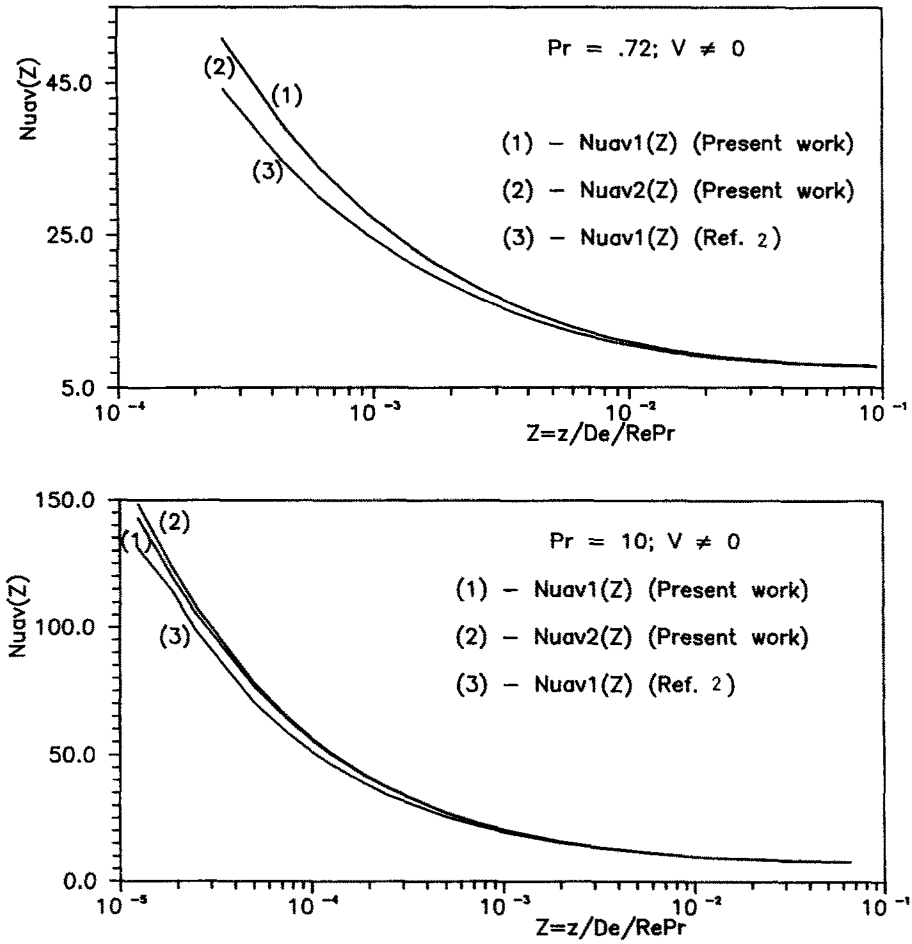

FIG. 2. Comparison of average Nusselt numbers from complete solution and from a finite-difference solution of full momentum and energy equations, in ref. [2] ($Pr = 0.72$).

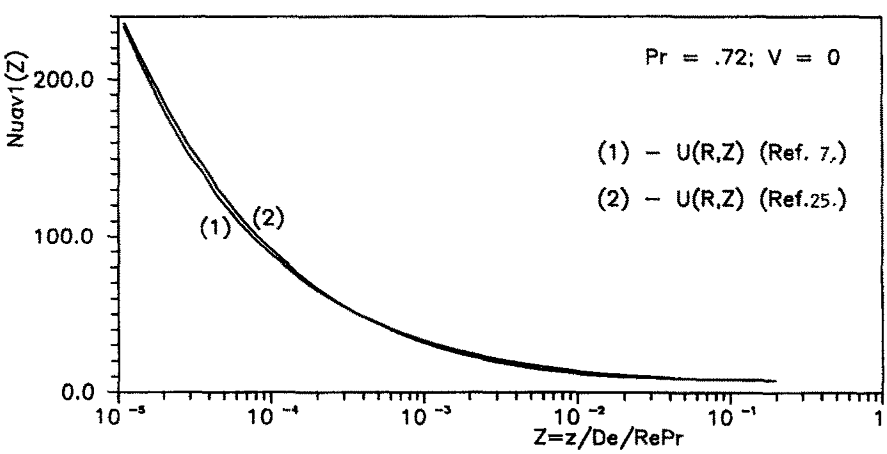

FIG. 3. Influence of axial velocity profile choice on average Nusselt number results ($Pr = 0.72$; $V = 0$).

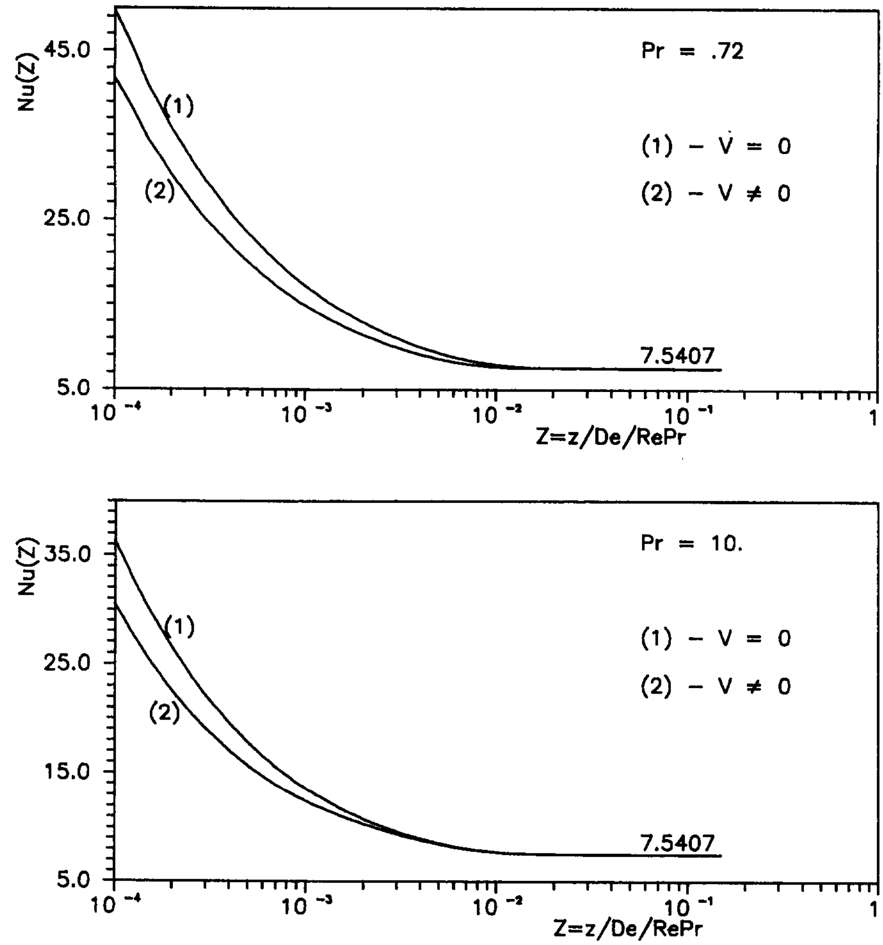

FIG. 4. Effect of neglecting normal convection term on local Nusselt number ($Pr = 0.72$).

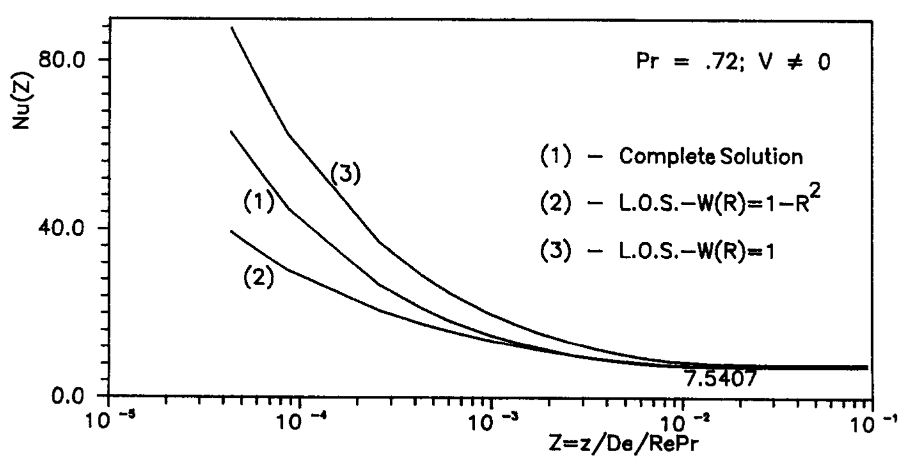

FIG. 5. Accuracy of local Nusselt number results from lowest order solution (L.O.S.) for two different choices of eigenvalue problem ($Pr = 0.72$).

the case of $Pr = 10$, which is not presented due to space limitations. This simple approximate solution is therefore only recommended for $Z \geqslant 5\times10^{-3}$, and with the appropriate choice of the Graetz-type eigenvalue problem.

CONCLUSIONS

The ideas in the generalized integral transform technique were successfully utilized in the hybrid analytical–numerical solution of simultaneously developing laminar channel flow, which represents an important class of diffusion–convection problems with non-separable equation coefficients. Explicit expressions for the velocity components were employed, as obtained from well-established linearization procedures, and the approximate heat transfer results are critically compared against those from purely numerical approaches, with a quite reasonable agreement.

The present success in extending the generalized integral transform technique provides additional experience and confidence towards the attempt of directly solving the full non-linear versions of such internal convection problems, by incorporating the also recently advanced ideas in the hybrid numerical–analytical solution of non-linear diffusion–convection problems [20, 24].

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APPENDIX. EXPRESSIONS FOR VELOCITY COMPONENTS

The linearization procedure introduced by Targ (25), is in fact a special case of the approach advanced by Sparrow *et al.* (7), and consists of approximating the inertia terms as

$$u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}\simeq u_0\frac{\partial u}{\partial z}. \tag{A1}$$

The resulting explicit expressions for the velocity components are then given by

$$U(R,Z)=\tfrac{3}{2}(1-R^2)+2\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\left[\frac{\cos(\alpha_n R)}{\cos\alpha_n}-1\right]e^{-\alpha_n^2 Z} \tag{A2}$$

$$V(R,Z)=2\sum_{n=1}^{\infty}\left[\frac{\sin(\alpha_n R)}{\alpha_n\cos\alpha_n}-R\right]e^{-\alpha_n^2 Z} \tag{A3}$$

where α_ns are the positive roots of the transcendental equation

$$\tan\alpha_n-\alpha_n=0. \tag{A4}$$

SOLUTIONS ANALYTIQUES DES DEVELOPPEMENTS SIMULTANES ASSOCIES A UN ECOULEMENT LAMINAIRE DANS UN CANAL A PLANS PARALLELES

Résumé—On étudie analytiquement les développements simultanés des vitesses et des températures pour un écoulement laminaire dans un canal entre plans parallèles, en adoptant une procédure linéarisée pour le problème de vitesse et en résolvant l'équation d'énergie découplée par la technique de transformation intégrale généralisée. Une solution complète est obtenue dans un large domaine de la coordonnée axiale à partir de l'évaluation numérique du système transformé intégral des équations différentielles. En outre des solutions explicites approchées sont données pour des estimations rapides dans le contexte des applications. On étudie plusieurs aspects tels que l'influence de la convection transverse, les effets de différents profils de vitesse, la convergence de la solution complète et la précision des solutions approchées.

ANALYTISCHE LÖSUNG FÜR DIE SICH SIMULTAN AUSBILDENDE LAMINARSTRÖMNG IN EINEM KANAL AUS PARALLELEN PLATTEN

Zusammenfassung—Die simultane Ausbildung der Geschwindigkeits- und Temperaturverteilung bei Laminarströmung in einem Kanal aus parallelen Platten wird analytisch untersucht. Dabei wird von einer Linearisierung des Geschwindigkeitsproblems ausgegangen, die Lösung der entkoppelten Energiebilanz erfolgt mit einem verallgemeinerten Integraltransformationsverfahren. Durch numerische Auswertung des so transformierten Systems gewöhnlicher Differentialgleichungen ergibt sich für einen weiten Bereich in axialer Richtung eine vollständige Lösung. Für eine schnelle Abschätzung bei der praktischen Anwendung werden zusätzlich explizite Näherungslösungen angeboten. Unterschiedliche Gesichtspunkte werden untersucht: Beispielsweise der Einfluß einer quergerichteten Konvektion, Effekte durch unterschiedliche Geschwindigkeitsprofile, Konvergenz der vollständigen Lösung sowie die Genauigkeit der Näherungslösungen.

АНАЛИТИЧЕСКИЕ РЕШЕНИЯ ЗАДАЧ РАЗВИВАЮЩЕГОСЯ ЛАМИНАРНОГО ТЕЧЕНИЯ В ПЛОСКО-ПАРАЛЛЕЛЬНЫХ КАНАЛАХ

Аннотация—Получено совместное аналитическое решение для определения скорости и температуры в плоско-параллельном канале, при этом в задаче для скорости использовалась линеаризация, а несвязанное уравнение энергии решалось методом обобщенных интегральных преобразований. На основе численного решения полученной системы обыкновенных дифференциальных уравнений найдено полное решение для широкого интервала вдоль аксиальной координаты. Приводятся также приближенные явные решения для быстрых оценок в конкретных приложениях. Исследуются такие вопросы как влияние поперечной конвекции и различных профилей скоростей, сходимость полного решения и точность приближенных решений.